(3) $f(x) = f(p) + f_1(p)(x_1 - p_1) + f_{11}(p_1 + sh, p_2, \cdots, p_n)(x_1 - p_1)^2/2.$

Here $h = x_1 - p_1$, $0 < s < 1$.

$$T(x) = f(p) + T_1(p)(x_1 - p_1). \tag{4}$$

Since $f(x) \geqq T(x)$, we find that

$$f_1(p) - T_1(p) + f_{11}(p_1 + sh, p_2, \cdots, p_n)(x_1 - p_1)/2 \geqq 0. \tag{5}$$

The quantity $f_1(p) - T_1(p)$ must be nonnegative, for otherwise we could choose $(x_1 - p_1)$ so small that (5) could not hold. (We note here $f_{11}(x) \geqq 0$ for $x \in D$ by hypothesis.) A similar consideration in the case where $p_1 > x_1$ shows that $f_1(p) - T_1(p) \leqq 0$. Hence $f_1(p) = T_1(p)$. In the same manner one can show that $f_i(p) = T_i(p)$, $i = 2, \cdots, n$. Thus $Q^*(x)$ and $T(x)$ are identical.

The idea for this note occurred to the author after hearing a lecture by Prof. Ranko Bojanic [1] on "best" one sided approximation in the case of functions of one variable.

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A Close Approximation Related to the Error Function*

By Roger G. Hart

A function has been found that closely approximates the integral function

$$F(x) = \int_x^\infty \exp(-t^2/2)\, dt$$

for all real values of x.

Let

$$P(x) = \frac{\exp(-x^2/2)}{x}\left[1 - \frac{(1 + bx^2)^{1/2}/(1 + ax^2)}{P_0 x + [P_0^2 x^2 + \exp(-x^2/2)(1 + bx^2)^{1/2}/(1 + ax^2)]^{1/2}}\right]$$
$$\equiv P_0 + x^{-1}\{\exp(-x^2/2) - [P_0^2 x^2 + \exp(-x^2/2)(1 + bx^2)^{1/2}/(1 + ax^2)]^{1/2}\},$$

where $P_0 = (\pi/2)^{1/2} \cong 1.253314137$,

$$a = \frac{1 + (1 - 2\pi^2 + 6\pi)^{1/2}}{2\pi} \cong .212023887,$$

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and

$$b = 2\pi a^2 \cong .282455120.$$

As may be readily verified, by inspection or substitution, $P(x)$ has the following properties in common with $F(x)$:

(1) For all real x, P is real, positive and finite.
(2) For all real x, dP/dx is real, negative and finite.
(3) For all real x, $P(x) + P(-x) = (2\pi)^{1/2}$.
(4) As $x \to 0$, $P(x) \to (\pi/2)^{1/2}$.
(5) As $x \to 0$, $dP/dx \to -1$.
(6) As $x \to \infty$, $P \to 0$ and $x \exp(x^2/2)P(x) \to 1$.
(7) As $x \to \infty$, $dP/dx \to 0$ and $(d/d(x^{-2}))[x \exp(x^2/2)P(x)] \to -1$.

Further resemblance between the two functions may be seen in the following table, where their values are compared for several x-values:

x	$F(x)$	$P(x)$
.1	1.1534806	1.1534812
.5	.77339	.77344
1.0	.3977	.3978
2.0	.05703	.05705
3.0	.0033837	.0033844
5.0	7.18529×10^{-7}	7.18532×10^{-7}
10.0	1.910014×10^{-23}	1.910008×10^{-23}

Values for $F(x)$ were obtained from [1].

The following table may be of help in assessing the possible applications of $P(x)$ approximations:

Integral I	*Approximation* A	*Magnitude and* x-*location of greatest absolute error*, $\lvert A-I\rvert_{max}$	*Magnitude and* x-*location of greatest relative error*, $\lvert (A-I)/I\rvert_{max}$
$\int_x^\infty \exp(-t^2/2)\,dt$	$P(x)$	.00013 for x near ± 1	.00055 for x near $+1.7$
$\int_{x_1}^{x_2} \exp(-t^2/2)\,dt$	$P(x_1) - P(x_2)$	.00027 for x_1 near -1, x_2 near $+1$	.0007 for small x-interval with x_1 and x_2 both near $+2$ or -2
$\sqrt{\frac{2}{\pi}} \int_0^x \exp(-t^2/2)\,dt$	$1 - \sqrt{\frac{2}{\pi}}\,P(x)$	.00011 for x near ± 1	.00016 for x near $\pm .9$
$\frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\,dt$ (the error function)	$1 - \sqrt{\frac{2}{\pi}}\,P(\sqrt{2}x)$	.00011 for x near $\pm .7$	.00016 for x near $\pm .6$

When compared with other approximations related to the error function,[2, 3, 4,5] $P(x)$ is seen to require more steps of computation but to mimic the integral function more faithfully over the whole x-range. However, the Hastings approximation[5] achieves, with its four arbitrary constants, a better fit in the range of low, positive x-values most often of interest, and it is therefore preferable for most applications. For $x > 2$, the Hastings approximation does not fit as well as $P(x)$. This leads to a somewhat paradoxical observation: While the greatest *absolute* error for $F(x)$ estimated from the Hastings approximation is only about one-fifth that obtained with $P(x)$, the greatest *relative* error with the latter is two orders of magnitude below those encountered with the Hastings approximation.

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Rational Approximations to the Solution of the Second Order Riccati Equation*

By Wyman Fair and Yudell L. Luke

I. Introduction. In a previous work Merkes and Scott [1] constructed continued fraction solutions to the first order Riccati equation by using a sequence of linear fractional transformations. Fair [2] utilized the τ-method, see the paper by Luke [3], to develop main diagonal Padé approximations to the solution of the first order Riccati equation with rational coefficients. Rational approximations are advantageous to study the behavior of the solutions in a global sense. That is, they are useful for evaluation of functional values in the complex plane including zeros and poles.

In this paper we develop continued fraction (and hence rational) approximations to the solution of a second order nonlinear equation which includes as special cases the equations treated in [1] and [2]. These approximations are obtained by using a sequence of linear transformations which leave the differential equation invariant, see Davis [4], and are presented in Section II. For an application, in Section III, the algorithm is applied to obtain approximations to Painlevé's first and second transcendents.

II. Development of the Rational Approximations. Consider the generalized second order Riccati equation

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